Polynomial and Rational Functions
\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \]

- \( f(x) = -5 \) Constant
- \( f(x) = 4x - 5 \) Linear Function
- \( f(x) = x^2 + 4x - 5 \) Quadratic Function
- \( f(x) = 2x^3 - 3x^2 + 6x^2 - 9 \) Higher-order Polynomial
- \( f(x) = \frac{2x^2 - 18}{x^3 - 27} \) Rational Function: Ratio of two polynomials

Polynomial and Rational Functions
\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \]

- \( f(x) = a_0 \) Horizontal Line
- \( f(x) = a_1x \) Tilted Line
- \( f(x) = a_2 x^2 \) Parabola
- \( f(x) = a_3 x^3 \) Cubic
- \( f(x) = a_n x^n \) to other powers
Combining powers of $x$.
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$

When $x = 0$, the constant term rules $f(x) = a_0$
When $x$ is far from 0, the highest power rules.
When $x$ is not too far from 0, the middle terms have their say.

End-behavior
When $x$ is far from 0, the highest power rules.

$f(x) = 3x^4 - 7x^2 + 3x$
$f(x) = 3x^4 - 7x^2 + 3x$
$f(x) \approx 3x^4$

Up as you move left
Up as you move right

Zeros and Roots

$x$ values for which $f(x) = 0$ are called "zeros" of $f$.
These $x$ values are called "roots" of $f(x) = 0$
They correspond to $x$-intercepts on the graph of $f(x)$

We can find zeros of $f(x)$ by factoring.

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$

$f(x) = a_n (x-r_1)(x-r_2) \cdots (x-r_n)$

$n$ factors, some may be the same
Some roots may be complex or repeated.
$f(x)$ has AT MOST $n$ $x$ intercepts

Repeated Roots

$x < c, (-) \times (-) \times (-) = (-)$

$f(x) = \frac{(x-c)(x-c)(x-c)(x-b)(x-a)}{x > c, (+) \times (+) \times (+) = (+)}$

$f(x)$ changes sign at $c$, so it must cut through $x$ axis

$x < c, (-) \times (-) = (+)$

$f(x) = \frac{(x-c)(x-c)(x-b)(x-a)}{x > c, (+) \times (+) = (+)}$

$f(x)$ doesn’t change sign at $c$, so it only touches $x$ axis

Repeated Roots

$f(x) = (x-2)(x-2)(x-5)(x+3)$
Turning Points

\[ f(x) = a_2x^2 + a_1x + a_0 \]

At most two \( x \)-intercepts
One turning point

Turning Points

\[ f(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \]

At most three \( x \)-intercepts
Two turning points

Turning Points

\[ f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \]

At most four \( x \)-intercepts
Three turning points

Turning Points

\[ f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 \]

At most \( n \) \( x \)-intercepts
At most \( n - 1 \) \( x \)-intercepts

Graphing Polynomials

1. Determine the end behavior
2. Find the \( x \) intercepts by solving \( f(x)=0 \). Roots of odd multiplicity cross, even touch.
3. Find the \( y \)-intercept by setting \( x = 0 \)
4. Check for and use any symmetry.
5. Make sure the number of turning points is \( n - 1 \) or less.

Graph \( f(x) = x^4 - 6x^3 + 9x^2 \)

End behavior is up and up.

\[ x^4 - 6x^3 + 9x^2 = 0 \]
\[ x^2(x^2 - 6x + 9) = 0 \]
\[ x^2(x - 3)^2 = 0 \]
\[ f(0) = (0)^4 - 6(0)^3 + 9(0)^2 = 0 \]
\[ f(-x) = (-x)^4 - 6(-x)^3 + 9(-x)^2 = x^4 + 6x^3 + 9x^2 \]
no symmetry