Rational Functions

\[ f(x) = \frac{p(x)}{q(x)} \]

Finding the domain

\[ f(x) = \frac{x - 6}{x^2 - 25} = \frac{x - 6}{(x - 5)(x + 5)} \]
\[
\text{dom } f = \{ x \mid x \neq 5 \text{ and } x \neq -5 \}
\]

\[ g(x) = \frac{2x - 3}{x^2 + 81} \]
\[
\text{dom } g = \text{all real numbers}
\]

\[ g(x) = \frac{2x - 3}{x^2 + 81} \]
\[ f(x) = \frac{x - 6}{x^2 - 25} \]

as \( x \to -5^- \)
\[ f(x) \to -\infty \]

as \( x \to -5^+ \)
\[ f(x) \to \infty \]

as \( x \to 5^- \)
\[ f(x) \to -\infty \]

as \( x \to 5^+ \)
\[ f(x) \to \infty \]

Finding Vertical Asymptotes

\[ g(x) = \frac{x}{(x - 3)(x + 2)} \]

as \( x \to \infty \)
\[ f(x) \to 0 \]

as \( x \to -\infty \)
\[ f(x) \to 0 \]
Finding Vertical Asymptotes

If \( f(x) = \frac{p(x)}{q(x)} \) is a rational function in which
\( p(x) \) and \( q(x) \) have no common factors and \( a \) is a zero
of \( q(x) \), then \( x = a \) is a vertical asymptote of the graph
of \( f \).

Finding Vertical Asymptotes

\[
\begin{align*}
h(x) &= \frac{x^3}{(x+7)(x+20)(x-5)} \\
&= \frac{x^3}{(x+7)(x+20)(x-5)} \\
&= \frac{x^3}{(x+7)(x+20)(x-5)} \\
&= \frac{x^3 + 6x + 9}{(x+3)(x-0)(x-5)} \\
&= \frac{x^3 + 6x + 9}{(x+3)(x-0)(x-5)} \\
&= \frac{(x+3)}{(x-0)(x-5)}
\end{align*}
\]

Finding Vertical Asymptotes

\[
\begin{align*}
k(x) &= \frac{x^3 - 4x + 3}{x^2 - 1} \\
&= \frac{(x-3)(x-1)}{(x+1)(x-1)} \\
&= \frac{(x-3)}{(x+1)}
\end{align*}
\]

Finding Vertical Asymptotes

\[
\begin{align*}
k(x) &= \frac{x^3 + 6x + 9}{(x+3)(x-0)(x-5)} \\
&= \frac{x^3 + 6x + 9}{(x+3)(x-0)(x-5)} \\
&= \frac{(x+3)}{(x-0)(x-5)}
\end{align*}
\]

Finding Vertical Asymptotes

\[
\begin{align*}
k(x) &= \frac{x^3 - 4x + 3}{x^2 - 1} \\
&= \frac{(x-3)(x-1)}{(x+1)(x-1)} \\
&= \frac{(x-3)}{(x+1)}
\end{align*}
\]

Horizontal Asymptotes

When \( x \) is far from 0,
\( x^a >> x^{a-1} >> ... >> x^2 >> x \) \( \gg 0 \)

\[
f(x) = \frac{3x^2}{2x^4} \text{ not important} \approx \frac{3x^2}{2x^4} = \frac{3}{2x}
\]

Anytime the degree of the denominator is
higher than the degree of the numerator, the
\( x \) axis will be the horizontal asymptote.
Horizontal Asymptotes
\[ f(x) = \frac{7x^4 + 2x^3 + 5x^2 - 2x + 6}{9x^4 + 2x^3 + 5x^2 - 2x + 6} \]
\[ f(x) = \frac{7}{9} \]
Anytime the degree of the denominator is the same as the degree of the numerator, the horizontal asymptote will be the line \( y = \frac{a_n}{b_n} \), where \( a_n \) is the leading order coefficient of the numerator and \( b_n \) is the leading order coefficient of the denominator.

Slant Asymptotes
\[ f(x) = \frac{x^2 + 5x^2 - 2x + 6}{x^2 - 4} \]
\[ f(x) = x + 5 \] as \( x \to \pm \infty \)

Slant Asymptotes
\[ f(x) = \frac{x^2 - x^3 + 3}{x^2 - 4} \]
\[ f(x) = x^2 + 3 \] as \( x \to \pm \infty \)

Slant Asymptotes
\[ f(x) = \frac{x^4 + 5x^2 - 2x + 6}{x^2 - 4} \]
\[ f(x) = x + 5 + \frac{2x + 26}{x^2 - 4} \] as \( x \to \pm \infty \)

Slant Asymptotes
\[ f(x) = \frac{x + 5}{x^2 - 4} \]
\[ f(x) = \frac{x + 5}{x^2 + 5x^2 - 2x + 6} \] as \( x \to \pm \infty \)

Slant Asymptotes
\[ f(x) = \frac{x + 5 + \frac{2x + 26}{x^2 - 4}}{x^2 + 5x^2 - 2x + 6} \]

Slant Asymptotes
\[ f(x) = \frac{x^2 + 3}{x^2 - 4} \]
\[ f(x) = \frac{x^2 + 3}{x^2 - 4} \] as \( x \to \pm \infty \)

Slant Asymptotes
\[ f(x) = \frac{x + 5 + \frac{2x + 26}{x^2 - 4}}{x^2 + 5x^2 - 2x + 6} \] as \( x \to \pm \infty \)

Anytime the degree of the numerator is one more than the degree of the numerator, the graph will approach a slanted line as \( x \) gets far from zero. The equation of that "slant asymptote" can be found from long division.
**x-intercepts**

$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0} = 0$

only happens when $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$

To find x-intercepts, find the zeros of the numerator (factors of the numerator that cancel factors of the denominator will disappear.)

**Graphing by Deduction**

0. Factor to get into lowest terms.
1. Check for symmetry
2. Find the y-intercept, if any (set $x = 0$)
3. Find any x-intercepts (set numerator = 0)
4. Find any vertical asymptotes (set denominator = 0)
5. Find the horizontal or slant asymptote (if any)
6. Plot at least one point between and beyond each x-intercept and vertical asymptote.
7. Plot the information from steps 1-6 to deduce the basic form of the graph.

**Graph by Deduction**

$f(x) = \frac{4x^2 - 1}{x^2 + 3}$

Symmetry about the $y$-axis

$y$-intercept at $(0, -\frac{1}{3})$

$x$-intercepts at $(-\frac{1}{2}, 0)$ and $(1, 0)$

No vertical asymptotes

Horizontal asymptote given by $y = \frac{-1}{3}$

$f(-1) = \frac{4(-1)^2 - 1}{(-1)^2 + 3} = \frac{3}{4}$

$f(1) = \frac{3}{4}$

**Graph by Deduction**

$f(x) = \frac{x^2 + 4}{x - 1}$

$f(-x) = \frac{(-x)^2 + 4}{-x - 1}$

No origin or $y$-axis symmetry

$f(0) = \frac{0^2 + 4}{0 - 1} = 4$ y-intercept at $(0, 4)$

$x^2 + 4 = 0$ no x-intercepts

Vertical asymptote is the line $x = 1$

Has slant asymptote $y = x + 1$
Graph by Deduction

\[ f(x) = \frac{x^3 + 4}{x - 1} \]

- No origin or \( y \)-axis symmetry
- \( y \)-intercept at \((0, -4)\)
- No \( x \)-intercepts
- Vertical asymptote is the line \( x = 1 \)
- Has slant asymptote \( y = x + 1 \)

\[
\begin{align*}
  f(5) &= \frac{5^2 + 1}{5 - 1} = \frac{26}{4} = 6.5 \\
  f(-5) &= \frac{(-5)^2 + 1}{-5 - 1} = \frac{26}{-6} = -4.33
\end{align*}
\]

Graph by Deduction

\[ f(x) = \frac{x + 6}{2x^2 - 5x + 2} \]

- No origin or \( y \)-axis symmetry
- \( y \)-intercept at \((0, 3)\)
- \( x \)-intercept at \((-2, 0)\)
- Vertical asymptotes given by \( x = 2 \) and \( x = \frac{1}{2} \)
- Horizontal asymptote is \( x \)-axis

\[
\begin{align*}
  f(0) &= \frac{0 + 6}{2(0)^2 - 5(0) + 2} = \frac{6}{2} = 3 \\
  x + 6 &= 0 \quad x = -6 \quad \text{- intercept at } (-6, 0) \\
  f(0) &= -3 \\
  f(-7) &= \frac{-1}{65} \quad f(1) = -7 \\
  f(3) &= \frac{9}{5}
\end{align*}
\]